PUTNAM PRACTICE SET 23

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Problem 1. Show that each positive integer can be written as a sum of integers of the form $2^a \cdot 3^b$ (for nonnegative integers a and b) with the property that no interger from the chosen sum divides a different integer from the sum.

Solution. We proceed by induction on the integer $n \geq 1$ which we want to express as a sum of integers of the form $2^a \cdot 3^b$; clearly, the statement holds for $n \in \{1, 2, 3\}$. Now, we assume the statement holds for all integers less than some number N (greater than 3) and next we show the same conclusion holds for N.

Now, if N is even, then we simply note that by our inductive hypothesis, the integer $\frac{N}{2}$ can be written as a sum of integers s_i of the form $2^a \cdot 3^b$ (none of those integers dividing a different integer from the same sum) and so, N is the sum of the corresponding integers $2s_i$ (with the same property).

Next, if N is odd, then we let m be the largest positive integer with the property that $3^m \leq N < 3^{m+1}$. We let

$$k := \frac{N - 3^m}{2};$$

clearly, if $k = 0$ then $n = 3^m$ and we are done. So, from now on, we assume $1 \leq k$; also, clearly, $k < N$. Now, by the induction hypothesis, we can write

$$k = \sum_{i=1}^{\ell} 2^{a_i} 3^{b_i},$$

with $a_i, b_i \geq 0$ and moreover, no integer from the above sum divides another one of those ℓ integers from our sum. We write then

$$N = 3^m + \sum_{i=1}^{\ell} 2^{a_i+1} 3^{b_i}$$

and we show next that no integer in the above sum divides a different integer from the above sum of $\ell + 1$ integers. Now, clearly no integer of the form $2^{a_i+1}3^{b_i}$ divides another integer of the same form (by our inductive hypothesis) and also, it cannot divide 3^m (since $a_i + 1 \geq 1$). So, the only possible obstruction to our desired conclusion would be if

$$3^m \mid 2^{a_i+1} 3^{b_i}$$

for some $1 \leq i \leq \ell$, i.e., if $m \leq b_i$. But then that would mean that $k \geq 2^{a_i} 3^{b_i} \geq 3^m$, i.e.,

$$3^m \leq \frac{N - 3^m}{2}$$

which means that $N \geq 3^{m+1}$, contradiction. This concludes our proof.

Problem 2. Let $n \in \mathbb{N}$ and let $P \in \mathbb{C}[z]$ be a polynomial of degree $2n$, all of whose roots have absolute value equal to 1. Let

$$g(z) := \frac{P(z)}{z^n}.$$

Prove that each solution for $g'(z) = 0$ (where g' is the derivative of g) has absolute value equal to 1.

Solution. We let $\omega_1, \dots, \omega_{2n}$ be all the roots of $P(z)$ (we allow for the possibility that some $\omega_i = \omega_j$ for $i \neq j$). We know that $|\omega_i| = 1$ for each $i = 1, \dots, 2n$. Now, if one of the solutions z_0 to the equation $g'(z) = 0$ is among the ω_i, then clearly also $|z_0| = 1$, as desired.

We have that $g'(z) = 0$ precisely when

$$P'(z) \cdot z^n - nz^{n-1} \cdot P(z) = 0$$

and $z \neq 0$ (note that $g(z) = P(z)/z^n$ and so, because $P(0) \neq 0$ because $\omega_i \neq 0$, then we cannot have that g and therefore g' is not defined at $z = 0$).

We let z_0 be a solution to the above equation and also assume $P(z_0) \neq 0$ (due to our assumption that $z \neq \omega_i$ for $i = 1, \dots, 2n$ because otherwise we would automatically have $|z_0| = 1$), then we can divide by $z_0^{n-1} \cdot P(z_0)$ and thus we get that

$$z_0 \cdot \frac{P'(z_0)}{P(z_0)} - n = 0.$$

An easy computation (using the product rule!) shows that

$$\frac{P'(z_0)}{P(z_0)} = \sum_{i=1}^{2n} \frac{1}{z_0 - \omega_i}.$$

So, after doubling the above equation, we get

$$0 = \left(\sum_{i=1}^{2n} \frac{2z_0}{z_0 - \omega_i}\right) - 2n = \sum_{i=1}^{2n} \left(\frac{2z_0}{z_0 - \omega_i} - 1\right) = \sum_{i=1}^{2n} \frac{z_0 + \omega_i}{z_0 - \omega_i}. \tag{1}$$

(Also, note that we assumed each $z - \omega_i$ is nonzero.)

After multiplying each fraction by $\bar{z_0} - \bar{\omega_i}$ and noting that $\omega_i \cdot \bar{\omega_i} = 1$, we get

$$0 = \sum_{i=1}^{2n} \frac{|z_0|^2 - 1 + (\bar{z_0}\omega_i - z_0\bar{\omega_i})}{|z_0 - \omega_i|^2}.$$

But $\bar{z_0}\omega_i - z_0\bar{\omega_i}$ is always of the form $i \cdot y$ for some real number y (i.e., it's a purely imaginary number, it has no real part). So, then taking the real part of the right hand side of (1) yields

$$0 = \sum_{i=1}^{2n} \frac{|z_0|^2 - 1}{|z_0 - \omega_i|^2} = \left(|z_0|^2 - 1\right) \cdot \sum_{i=1}^{2n} \frac{1}{|z_0 - \omega_i|^2},$$

which forces $|z_0| = 1$ because the above sum from the right hand side is always positive.

Problem 3. Let A be an N-by-N matrix with the property that each one of its entries is equal to 1 or -1 and also satisfying that $A \cdot A^t = N \cdot \mathrm{id}_N$ (where id_N is the N-by-N identity matrix). Assume there exists an a-by-b submatrix of A whose entries are all equal to 1. Prove that $ab \leq N$.

Solution. We let $v_1, \dots, v_a$ be the a rows in the matrix A with the property that the a-by-b submatrix containing only the entries equal to 1 is part of the a-by-N submatrix of A formed by the rows $v_1, \dots, v_a$. We let

$$w := \sum_{i=1}^{a} v_i$$

and denote by $|w|^2 = w \cdot w^t$ the length of this vector. We compute

$$w \cdot w^t = \left(\sum_{i=1}^{a} v_i\right) \cdot \left(\sum_{j=1}^{a} v_i^t\right) = \sum_{1 \le i,j \le a} v_i \cdot v_j^t.$$

Finally, noting that $v_i \cdot v_j^t = 0$ for $i \neq j$, while

$$v_i \cdot v_i^t = N \text{ for each } i = 1, \dots, a,$$

we get $|w|^2 = N \cdot a$. On the other hand, we know that the vector w contains b entries each one of them equal to a. Therefore,

$$|w|^2 \ge b \cdot a^2,$$

which combined with the fact that $|w|^2 = Na$ yields the desired inequality $ab \le N$.

Problem 4. Evaluate

$$\int_0^1 \frac{\ln(x+1)}{x^2+1} \mathrm{dx}$$

Solution. We make the substitution $x = \tan(t)$ and so, $\mathrm{dx} = \sec^2(t) \cdot \mathrm{dt}$, while the bounds of integration change to $t = 0$ and $t = \frac{\pi}{4}$ and so, our integral I equals now

$$\int_0^{\frac{\pi}{4}} \frac{\ln(\tan(t)+1)}{\tan^2(t)+1} \cdot \sec^2(t)\mathrm{dt} = \int_0^{\frac{\pi}{4}} \ln(\tan(t)+1)\mathrm{dt} = \int_0^{\frac{\pi}{4}} \ln(\sin(t)+\cos(t)) - \ln(\cos(t))\mathrm{dt}$$

Now, we use the identity

$$\sin(t) + \cos(t) = \cos\left(\frac{\pi}{2} - t\right) + \cos(t) = 2\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4} - t\right) = \sqrt{2} \cdot \cos\left(\frac{\pi}{4} - t\right)$$

and so, we get that our integral I equals

$$\int_0^{\frac{\pi}{4}} \ln(\sqrt{2}) + \ln\left(\cos\left(\frac{\pi}{4} - t\right)\right) - \cos(t)\mathrm{dt} = \frac{\pi}{4} \cdot \ln(\sqrt{2}) + \int_{t=0}^{\frac{\pi}{4}} \ln\left(\cos\left(\frac{\pi}{4} - t\right)\right)\mathrm{dt} - \int_{t=0}^{\frac{\pi}{4}} \ln(\cos(t))\mathrm{dt}.$$

On the other hand, using the substitution $t = \frac{\pi}{4} - u$, we get that

$$\int_0^{\frac{\pi}{4}} \ln\left(\cos\left(\frac{\pi}{4} - t\right)\right)\mathrm{dt} = \int_0^{\frac{\pi}{4}} \ln(\cos(u))\mathrm{du}$$

and so, our integral I equals

$$\frac{\pi}{4} \cdot \ln(\sqrt{2}) = \frac{\pi \ln(2)}{8}.$$